PRODUCTION OF BURSTS BY MESON AND ITS DEPENDENCE ON THE MESON SPIN

By S. K. CHAKRABARTY

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ABSTRACT. An analytic expression has been derived for the calculation of the frequency of bursts containing N particles or more than N particles produced by a meson. Three different values for the meson spin, *viz.*, 0, ½ and 1, have been taken into consideration and the results thus obtained have been compared with the experimental results of Schein and Gill. It is observed that for a meson, spin 0 or ½ is not a possibility, whereas spin 1 is quite probable, provided the effect of the radiation damping be taken into consideration in processes involving mesons. The reasons for arriving at a conclusion just opposite to that of Christy and Kusaka have been discussed. It has also been shown from general considerations that for large bursts the radiation process is the predominant one, when the knock-on process produces only an insignificant contribution.

The production of large bursts under thick layers of a material has been observed and their frequency has been estimated by various authors. Bhabha[1] (1938) tried to explain these as a consequence of a knock-on collision, in which a meson while passing through a thick layer of a material makes a very close collision with an atomic electron and knocks out the electron, which subsequently produces a large number of particles by the ordinary cascade process. His calculations are based on the assumption that the meson has a spin of half a unit and thus can be described by the Dirac equation. Later on, Bhabha, Carmichael and Chou[2] (1939) have modified these calculations on the assumption that the mesons have a spin of one unit and are described by the equations given by Proca[3] (1936). Large bursts are naturally produced by high-energy mesons, and bursts containing more than 100 particles can only be produced by mesons having energies more than 10^{10} e.v. But in this energy range, the cross-sections for the production by a meson of a secondary knock-on electron is different for different spins, and consequently it is believed that a comparison of the theoretical results on the frequency of the burst production, containing a large number of particles, will provide a good test of the meson theory. High-energy mesons have also a large probability of radiating high-energy quanta, while passing the electric field of the nucleus (*Bremsstrahlung*), which again will produce showers by the ordinary cascade process. Bhabha[1] (1938) has shown that if the meson has a spin of half a unit, the number of electrons produced through the radiation process, for a meson of mass 100 times or more than that of an electron, is negligible in comparison with that produced by the knock-on process, but similar calculations were not extended when the meson has a spin of one unit. Moreover, as will be evident in the following section, such a conclusion is not valid for large bursts. Christy and Kusaka[4] (1941) have made these calculations, both for the knock-on process and the radiation process, and taking for the meson various possible spins, *viz.*, zero, half and one unit. Finally they have compared their results with the experimental results of Schein and Gill[5] (1939) and have derived the conclusion that mesons, at least those observed in cosmic rays, cannot have spin 1, but may have a spin zero or half, and preferably zero.

The most serious uncertainty in these calculations is the lack of knowledge of the fluctuations. Christy and Kusaka have assumed Furry's fluctuation formula while Bhabha and others have taken the Poisson distribution. But it is well known that Furry's fluctuation formula is justified only for very small thickness of the material traversed, and is not valid for large thicknesses. But for small thicknesses Furry's formula deviates but little from the Poisson formula. Consequently in large thickness of the material, which is more important in the burst production, it is doubtful whether Furry's fluctuation formula can be used. Another uncertainty in these calculations of the burst frequency arises due to the rough approximations usually made in the Cascade theory, since in the production of the bursts cascade multiplication plays in the end a decisive part. Christy and Kusaka have taken, for the average number of particles produced in the cascade process, an approximate form which very roughly fits with the results of Serber's[6] (1938) calculations. But this expression for the average is in error for various reasons.* Moreover for reasons already pointed out (Chakrabarty[7], 1942) the result of Serber is seriously open to doubts. As the conclusion derived from the results obtained by Christy and Kusaka has far reaching effects, it is desirable to see how far these approximations may affect their final result.

In the present paper we shall calculate the frequency of bursts produced by a meson in a thick layer of a material, assuming that the fluctuation around the average obey a Poisson distribution and using the accurate results of the Cascade theory obtained previously.[8] A comparison of the results of calculations of the present paper with the experimental results of Schein and Gill[5] (1939) will also be made at the end.

* The expression for S_{AV} giving the average number of particles produced by a primary of energy E at a depth x (in shower units of length), as taken by Christy and Kusaka, has always a maximum at $x=7$ and the corresponding maximum value of S_{AV} is $E/9\beta$, whatever the value of E may be. Apart from the fact that the position of the maximum for S_{AV} will change with the change of E, the value of S_{AV} at the maximum is in excess by a factor ranging from 1.5 to 2.1 as E increases from 10^{10} e.v. to 10^{12} e.v., when compared with the accurate figures for S_{AV} obtained by us (Bhabha and Chakrabarty, 1942)[8]. This is primarily due to the defects in the analysis of Serber, as has already been shown in a previous paper. These considerations show that neither in the position nor in the height of the maximum the approximation is valid, and it is difficult to estimate how much these approximations will affect the final results.

To investigate the production of showers by heavy particles, two distinct processes are to be considered. (1) The meson may produce a very fast secondary electron by direct collision which will subsequently produce a shower by cascade multiplication. (2) The meson may radiate a high-energy quantum which ultimately produces the shower. The cross-sections for these processes, taking different spins and magnetic moments, have been given by various authors [Bhabha[1] (1938), Massey and Corben[9] (1939), Corben and Schwinger[10] (1940), Christy and Kusaka[4] (1941), Bothe and Wilson[11] (1940), Wilson[12] (1941)].

The differential effective cross-section $Q(W, E_0)dE_0$ for the production by a meson of energy W, of a secondary electron or a quantum, by either of the two processes mentioned above, and having energies lying between E_0 and $E_0 + dE_0$, are given by the following expressions, depending on the spin of the meson. Since we are concerned with large bursts and consequently with highly energetic mesons, we can take $W >> Mc^2$, where M is the mass of the meson. With this approximation we have for the knock-on process; and

(*i*) Spin 0, magnetic moment 0

$$Q_c(W,E_0)dE_0 = 2\pi r_0^2 mc^2 Z[1 - E_0/E_{0m}]E_0^{-2}dE_0, \qquad \dots \quad (1a)$$

(*ii*) Spin $\frac{1}{2}$, magnetic moment $e\hbar/2Mc$,*

$$Q_c(W,E_0)dE_0 = 2\pi r_0^2 mc^2 Z[1 - (E_0/E_{0m}) + \tfrac{1}{2}(E_0^2/W^2)]E_0^{-2}dE_0, \qquad \dots \quad (1b)$$

(*iii*) Spin 1, magnetic moment $e\hbar/2Mc$,

$$Q_c(W,E_0)dE_0 = 2\pi r_0^2 mc^2 Z\left[1 - \frac{E_0}{E_{0m}} + \frac{1}{3}\frac{E_0^2}{W^2} + \frac{1}{3}\frac{mE_0}{M^2c^2}\left\{1 - \frac{E_0}{E_{0m}} + \frac{1}{2}\frac{E_0^2}{W^2}\right\}\right]\frac{dE_0}{E_0^2}, \qquad \dots \quad (1c)$$

where $r_0 = e^2/mc^2$ and E_{0m} is the maximum energy which can be communicated to an electron in a free collision, and is given by

$$E_{0m} = W[1 + Mc^2/(2mW)]^{-1}. \qquad \dots \quad (2)$$

Similarly the expressions for $Q(W,E_0)dE_0$ for the radiation process are given by the following expressions.

(*i*) Spin 0, magnetic moment 0,

$$Q_r(W,E_0)dE_0 = \frac{16}{3}\cdot\frac{Z^2 r_0^2}{137}\left(\frac{m}{M}\right)^2\left(\frac{W - E_0}{E_0}\right)\left[\log\frac{2W(W - E_0)}{Mc^2 E_0} - \frac{1}{2}\right]\frac{dE_0}{W}, \qquad (3a)$$

(*ii*) Spin $\frac{1}{2}$, magnetic moment $e\hbar/2Mc$,

$$Q_r(W, E_0)dE_0 = \frac{16}{3}\cdot\frac{Z^2 r_0^2}{137}\left(\frac{m}{M}\right)^2\left[\frac{W - E_0}{E_0} + \frac{3}{4}\frac{E_0}{W}\right] \times \left[\log\frac{2W(W - E_0)}{Mc^2 E_0} - \frac{1}{2}\right]\frac{dE_0}{W}, \qquad \dots \quad (3b)$$

when the screening of the atomic nuclei is neglected, and

* $\hbar$ denotes Planck's constant divided by 2π.

$$Q_r(W, E_0)dE_0 = \frac{16}{3} \cdot \frac{Z^2 r_0^2}{137} \left(\frac{m}{M}\right)^2 \log\left(183 \frac{M}{m} Z^{-\frac{1}{3}}\right) \left[\frac{W - E_0}{E_0} + \frac{3}{4} \frac{E_0}{W}\right] \frac{dE_0}{W}, \quad (3c)$$

when the screening is complete.

(*iii*) Spin 1,* magnetic moment $e\hbar/2Mc$,

$$Q_r(W, E_0)dE_0 = \frac{1}{12} \cdot \frac{Z^2 r_0^2}{137} \left(\frac{m}{M}\right)^2 \frac{W}{Mc^2}\left[2 - 2\frac{E_0}{W} + 7\frac{E_0^2}{W^2}\right] \frac{dE_0}{W}, \quad \ldots \quad (3d)$$

when the effect of radiation damping is neglected. If the effect of radiation damping be taken into consideration, then as given by Wilson[12] (1941) we have for spin 1,

$$Q_r (W, E_0)dE_0 = \frac{\pi}{6} \cdot \left(3^{-\frac{7}{6}} a^{-\frac{2}{3}}\right) \cdot \frac{Z^2 r_0^2}{137} \left(\frac{m}{M}\right)^2 \log\left(137 \frac{M}{m} Z^{-\frac{1}{3}}\right)$$

$$\times \left[2 - 2\frac{E_0}{W} + 7\frac{E_0^2}{W^2}\right] \frac{dE_0}{W}. \quad \ldots \quad (3e)$$

We shall first calculate the probability P(N), of a meson being accompanied by N electrons and positrons after passing through a layer of some substance of thickness t and of atomic number Z. If $\bar{N}(y_0, t)$ be the average number of particles produced at a depth t in characteristic units, defined previously,[8] produced by a primary particle or quantum of energy β exp. y_0, then as shown by Bhabha and Heitler[14] (1937) the probability of N particles appearing instead of $\bar{N}$ by a fluctuation is given by

$$\exp.(-\bar{N}).\bar{N}^N/\Gamma(N+1). \quad \ldots \quad (4)$$

Following Bhabha if we neglect the small variation in the energy of the heavy particle, we get

$$P(N) = l\sigma \int_0^\epsilon Q(W, E_0) J_N(y_0) dE_0, \quad \ldots \quad (5)$$

where $$J_N(y_0) = \int_0^t dt' \exp.(-\bar{N})\{\bar{N}(y_0, t')\}^N/\Gamma(N+1), \quad \ldots \quad (6)$$

where $Q(W, E_0)$ is the cross-section given by equations (1) or (3), according as one considers the knock-on process or the radiation process, for which the values of ϵ will be E_{0m} or W respectively. σ represents the number of atoms per cubic centimetre of the substance, and

$$y_0^* = \log_e(E_0/\beta), \quad \ldots \quad (7)$$

* The expression given by Christy and Kusaka, differs from that given by Booth and Wilson in having an additional factor $\pi/(5Mc^2Z^{\frac{1}{3}})$ and also in having further terms varying as log W. These differences arise due to the difference in the nature of the approximations made in the limits of the impact parameter. The logarithmic terms will, however, have no appreciable effect in the calculations of the burst frequency.

where β is the critical energy (or the mean collision loss in the characteristic unit) for the substance of the material and has been defined in a previous paper.[8] Following Bhabha, the upper limits of the integral in (6) can be extended to ∞ when we are concerned with large thickness of the material. We therefore have

$$J_N(y_0) \simeq \int_0^\infty dt \exp.(-\bar{N})\,\bar{N}^N/\Gamma(N+1). \qquad \ldots \quad (6a)$$

From the results of the cascade theory (Bhabha and Chakrabarty, 1942)[8] we can easily calculate the values of $\bar{N}$ for different values of y_0 and t. When $\bar{N}(y_0, t)$ for any given y_0 is plotted against t it is observed that the curves are of the skew type of a Gaussian curve. It is observed that $\bar{N}$ can be obtained empirically by the process of curve fitting and the best fit is given by the following equations:

$$\bar{N}=N_m \exp.\{-a_1 t_m(1-t/t_m)^2\}, \qquad \text{when} \quad t \leq t_m \qquad \ldots \quad (8a)$$

$$\bar{N}=N_m \exp.\{-a_2 t_m(t/t_m-1)^2\}, \qquad \text{when} \quad t \geq t_m \qquad \ldots \quad (8b)$$

where N_m is the maximum value of $\bar{N}$ and t_m the value of t at this point for a given y_0 and were deduced accurately in a previous paper.[8] They are given by*

$$t_m=1.01y_0-1.92 \qquad \ldots \quad (9a)$$

and
$$N_m=0.169(y_0-1.90)^{-\frac{1}{2}} \exp.\, y_0 \simeq 0.169\, t_m^{-\frac{1}{2}}. \exp.\, y_0, \qquad \ldots \quad (9b)$$

with $a_1=0.70$ and $a_2=0.17$.

Except for very small values of t (~ 1) and again when $t >> t_m$ the fit of the empirical curves with the actual curves is very good, and in any case even for small values of t, the error introduced in taking (8) is well within the limits of error already existing in the calculations of the Cascade theory.

From (6a) and (8) we thus have,

$$J_N(y_0)=\int_0^\infty dt\, e^{-\bar{N}}\, \bar{N}^N \,/\, \Gamma(N+1)$$

$$=\frac{N_m^N\, t_m}{\Gamma(N+1)} \cdot \int_0^\infty \sum_{r=0}^{\infty} (-1)^r \cdot \frac{N_m}{\Gamma(r+1)}$$
$$\times\left[\exp.\{-(N+r)a_1 t_m z^2\}+\exp.\{-(N+r)a_2 t_m z^2\}\right]dz$$

$$=\tfrac{1}{2}\sqrt{\pi}\left(a_1^{-\frac{1}{2}}+a_2^{-\frac{1}{2}}\right)\left(N_m^N\, t_m^{\frac{1}{2}} \,/\, \Gamma(N+1)\right)\sum_0^\infty(-1)^r . N_m^r/\{\Gamma(r+1)(N+r)^{\frac{1}{2}}\},$$

$$=3.21\, t_m^{\frac{1}{2}} . \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)N_m^{N-z}}{\Gamma(N+1)\sqrt{(N-z)}} . dz, \qquad \ldots \quad (10)$$

* These values are slightly modified when the variations of the cross-sections for radiation and pair-creation processes with energy and also with the nature of the material is considered in the Cascade theory. This has been shown in a different paper.

where c is any real number greater than zero. The integral in (10) can be evaluated by the saddle point method. It can easily be shown that so long as $N_m < N$, the saddle point lies in the neighbourhood of N_m and the corresponding value of $J_N(y_0)$ tends to zero, but when $N_m > N$ the saddle point lies in the neighbourhood of N, but is always less than N and in that case the corresponding values of $J_N(y_0)$ is finite, has a maximum when N_m is slightly greater than N, and for larger values of N_m (*i.e.*, of y_0) the values of $J_N(y_0)$ slowly diminishes. This behaviour of $J_N(y_0)$ will be apparent from Fig. 1, in which the values of $J_N(y_0)$ for different values of N have been plotted against y_0. Bhabha[1] (1938) has also explained such a behaviour of $J_N(y_0)$ from geometrical considerations. Consequently, as has already been mentioned by Bhabha, it is now clear that for a rough estimate one

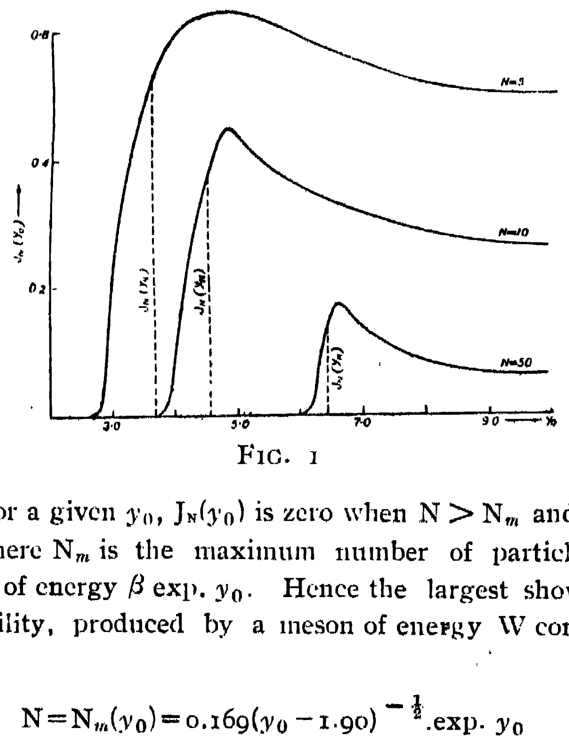


Fig. 1

can assume that for a given y_0, $J_N(y_0)$ is zero when $N > N_m$ and remains constant when $N \leq N_m$, where N_m is the maximum number of particles produced by a particle or quanta of energy β exp. y_0. Hence the largest shower which occurs with any probability, produced by a meson of energy W contains N particles, where

$$N = N_m(y_0) = 0.169(y_0 - 1.90)^{-\frac{1}{2}}.\text{exp. } y_0 \qquad \ldots \quad (11)$$

and $y_0 = \log (E_{om}/\beta)$ or $\log (W/\beta)$, depending on whether the shower is produced through a knock-on process or a radiation process respectively. A similar expression has also been obtained by Bhabha, where he has considered only those particles in the shower having energies greater than the critical energy. Lovell[13] (1939) tried to obtain a similar expression, taking into consideration the particles having energies less than the critical energy. But his values are too large, roughly by a factor four.

From (5) and (10) we can now proceed to calculate the values of P(N) for different values of N using for $Q(W, E_0)$ different expressions given by (1) and (3). All the integrations occurring cannot be evaluated analytically, and we have to calculate them numerically. But this is not the quantity which is of direct physical interest, and in order to compare the theoretical results with experiments we have to integrate this probability over the spectrum of the meson which is incident on the layer of material under consideration. This can be done as in the next section. But to determine only the order of P(N) one can follow Bhabha, and take $J_N(y_0)=0$ when $y_0 < y_N$ and $J_N(y_0)=J_N(y_N)$ when $y_0 \geq y_N$ where y_N is that value of y_0 which makes $N_m=N$.* A comparison of the values of $J_N(y_0)$ derived in the present paper with the corresponding values obtained by Bhabha shows that the values of $J_N(y_0)$ taken by him are nearly twice those of the values obtained in the present paper. Bhabha has introduced a factor c to explain the behaviour of $J_N(y_0)$ which he initially stated to be of the order of unity. The argument given above shows that it can be taken to be unity. This probably explains why the values calculated by Bhabha for Σ NP(N) were roughly twice the calculated average number of particles accompanying the heavy particles. Consequently, the value of c will be 1 and not 2 as has been finally suggested by Bhabha. Lovell has taken $c=0.25$ which naturally makes his result too large by a factor four.

To compare with the experimental results of Schein and Gill and others and also with the theoretical results of Christy and Kusaka it is necessary to calculate the probability of a meson producing *more than* N particles. Bhabha and Heitler[11] (1937) have shown that if the fluctuation obeys the Poisson law, then the probability of getting *more than* N particles in a shower, when the average is $\bar{N}$ is $W(N+1, \bar{N})$,

where
$$W(N+1, N) = \frac{N^{N+}}{\Gamma(N+1)} \cdot \int_0^1 z^N \exp.(-\bar{N}z)dz.$$

Consequently, the probability $P(N_>, W)$ of a meson of energy W emerging accompanied by a shower containing *more than* N particles from an infinitely thick layer of a substance is given by

$$P(N_>, W) = l\sigma \int_0^{\epsilon} Q(W, E_0) J_{N>}(y_0) dE_0, \qquad \dots \quad (12)$$

where
$$J_{N>}(y_0) = \frac{1}{\Gamma(N+1)} \int_0^{\infty} \bar{N}^{N+1}\, dt \int_0^1 z^N \exp.(-\bar{N}z)dz. \qquad \dots \quad (13)$$

Substituting the values of $\bar{N}$ from equation (8) and interchanging the order of integration in (13) we have

$$J_{N>}(y_0) = \frac{N_m^{N+1}}{\Gamma(N+1)} \cdot \tfrac{1}{4}\sqrt{\pi}\,(a_1^{-\frac{1}{2}} + a_2^{-\frac{1}{2}}) t_m^{\frac{1}{2}} \sum_0^{\infty} (-1)^r N_m^r, \{\Gamma(r+1), (N+r+1)^{+\frac{3}{2}}\}.$$

$$= 3.21. \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z) N_m^{N-z+1}\, t_m^{\frac{1}{2}}}{\Gamma(N+1)(N-z+1)^{\frac{3}{2}}} dz. \qquad \dots \quad (14)$$

* This approximation is no longer necessary, since the analytical expression for $J_N(y_0)$ as given by (10) can now be used for the calculations of P(N).

If $S(W)dW$ be the number of mesons, having energies between W and $W+dW$, incident on the layer of the material under consideration, then the total probability of getting a shower containing *more than* N particles below an "infinitely thick"* layer of the material is given by

$$B(N)=l\sigma\int_0^\infty S(W)dW\int_0^\epsilon Q(W.E_0)J_{N>}(y_0)dE_0$$
$$=3.21\ l\sigma.\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z)(.169)^{N-z+1}}{\Gamma(N+1)(N-z+1)^{\frac{3}{2}}}\,dz\int_0^\infty S(W)dW$$
$$\times\int_0^\epsilon Q(W,E_0)(y_0-\zeta)^{-\frac{1}{2}(N-z)}\exp.(N-z+1)y_0.dE_0 \quad \ldots \quad (15)$$

where $\zeta=1.90$

As in the case of $J_N(y_0)$ it can also be shown here that for all values of y_0 for which $N_m<N$, $J_{N>}(y_0)\simeq 0$, and for other values of y_0, $J_{N>}(y_0)$ gradually increases as y_0 increases. Consequently the lower limit of the E_0 integration and hence also of the W integration can be taken to be E_N or any other value less than E_N whichever is convenient, where E_N is that value of E_0, corresponding to which $N_m=N$. For convenience in the evaluation of the integrals in (15) we shall henceforth take the lower limit of the E_0 integration as β exp. ζ. This is justified so long as the value of N is greater than 3.

Following Euler and Heisenberg[15] (1938) we have assumed that

$$S(W)dW=\delta I_0\frac{(2\times10^9.e.v)^\delta}{(\epsilon T+W)^{\delta+1}}\,dW, \quad \ldots \quad (16)$$

where $\delta=1.87$ and $\epsilon T=2.10^9$ e.v. at sea-level. Also from the analysis of Street and Woodward[16] (1934) and Blackett[17] (1938) we assume that the total meson intensity at sea-level is 0.01 per cm.^{2} per sec. per unit solid angle. Hence from equations (1), (3), (15) and (16) we can now calculate the number of bursts of size *greater than* N per cm.^{2} per sec. due to any particular cross-section. This is what Christy and Kusaka has calculated but with a different expression for $\bar{N}$ and a different form for fluctuation.

We shall first consider those bursts produced by a meson radiating a hard quantum which ultimately produce the burst through the usual cascade process and so take for $Q(W, E_0)$ its values given by equations (3). We therefore have

$$B(N)=\delta I_0(2\times10^9\text{ e.v.})^\delta.(3.21\ l\sigma).\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z)(0.169)^{N-z+1}}{\Gamma(N+1)(N-z+1)^{\frac{3}{2}}}\,dz$$
$$\times\int_{\beta.\zeta}^\infty\frac{\exp.(N-z+1)y_0}{(y_0-\zeta)^{\frac{1}{2}(N-z)}}\,dE_0\int_{E_0}^\infty Q(W,E_0).\frac{dW}{(\epsilon T+W)^{\delta+1}}. \quad \ldots \quad (17)$$

* In the sense used by Bhabha (1938).

For the caculations of large bursts for which $N \geq 100$ it is necessary to consider only those mesons for which $W > E_N$ ($\approx \beta c^7$). So that as long as $N \geq 100$, for convenience in integration we can in (17) take $W^{-(\delta+1)}$ instead of $(\epsilon T+W)^{-(\delta+1)}$, since the error thus introduced will be negligible. It may be noted here that as the number of incident mesons decreases rapidly as W increases (when $W > \epsilon T$), and on the other hand $P(N_>, W)$ tends to zero when $W < E_N$ and is finite and gradually increases as W increases, the combined effect of S(W) and $P(N_>, W)$ will show that the major part of B(N) is contributed by the values of W in the neighbourhood of E_N. This has also been shown graphically by Lovell. Consequently, any error either in the value of S(W) or $P(N_>, W)$ in the neighbourhood of $W = E_N$ will produce an appreciable effect in the calculations of B(N).

It will be shown in the appendix, that the values of B(N) for different values of $Q(W, E_0)$ as given by (3) can be written in the following form :—

(*i*) Spin 0,

$$B(N) = A\ \exp.(-\delta\zeta).\{(K_0 + \zeta L_0) f_1(N, \delta) + L_0 f_2(N, \delta)\}. \quad \dots \quad (18a)$$

(*ii*) Spin half (screening neglected),

$$B(N) = A\ \exp.(-\delta\zeta).\{(K_{\frac{1}{2}} + \zeta L_{\frac{1}{2}}) f_1(N, \delta) + L_{\frac{1}{2}} f_2(N, \delta)\}. \quad \dots \quad (18b)$$

(*iii*) Spin half (screening complete),

$$B(N) = AK'_{\frac{1}{2}}\ \exp.(-\delta\zeta).f_1(N, \delta). \quad \dots \quad (18c)$$

(*iv*) Spin 1 (neglecting radiation damping),

$$B(N) = AK_1\ \exp.(-\delta\zeta).\exp.\zeta.f_3(N, \delta). \quad \dots \quad (18d)$$

(*v*) Spin 1 (considering the effect of radiation damping),

$$B(N) = AK'_1\ \exp.(-\delta\zeta).f_1(N, \delta), \quad \dots \quad (18e)$$

where
$$A = \delta I_0 (3.21 l\sigma) \frac{Z^2 r_0^2}{137} \left(\frac{m}{M}\right)^2 \left(\frac{2 \times 10^9 . \text{e.v.}}{\beta}\right)^\delta. \quad \dots \quad (19a)$$

$$K_0 = L_0 [\{\log(2\beta/Mc^2) - \tfrac{1}{2}\} + 2\delta^{-1}(\delta+1)^{-1}(2\delta+1) + \delta\psi(\delta+2) - (\delta+1)\psi(\delta+1) - \gamma], \quad \dots \quad (19b)$$

$$L_0 = \frac{16}{3}\delta^{-1}(\delta+1)^{-1}, \quad \dots \quad (19c)$$

$$K_{\frac{1}{2}} = K_0 + 4[(\delta+1)^{-1}\{\log(2\beta/Mc^2) - \tfrac{1}{2}\} + 2(\delta+2)^{-2} - (\delta+2)^{-1}\{\psi(\delta+3)+\gamma\}], \quad (19d)$$

$$L_{\frac{1}{2}} = L_0 + 4(\delta+1)^{-1}, \quad \dots \quad (19e)$$

where $\psi(\delta) = \frac{d}{d\delta}\log_e \Gamma(\delta)$ and γ is the Euler-Mascheroni constant.

$$K'_{\frac{1}{2}} = \frac{16}{3}\log\left(183\frac{M}{m}Z^{-\frac{1}{3}}\right)\left[\delta^{-1}(\delta+1)^{-1} + \frac{3}{4}(\delta+2)^{-1}\right], \quad \dots \quad (19f)$$

$$K_1 = \frac{1}{12}.\frac{\beta}{Mc^2}\left[2\delta^{-1}(\delta+1)^{-1} + 7(\delta+2)^{-1}\right], \quad \dots \quad (19g)$$

$$K'_1 = \frac{\pi}{6} \cdot \left(3^{-\frac{7}{6}} a^{-\frac{2}{3}}\right) \log \left(137 \frac{M}{m} Z^{-\frac{1}{3}}\right) \left| 2(\delta+1)^{-1}(\delta+2)^{-1} + 7(\delta+3)^{-1} \right|, \tag{19h}$$

and

$$f_1(N, \delta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(N, z) \Gamma \left(1 - \frac{N-z}{2}\right) (\delta + z - N - 1)^{-\frac{1}{2}(2-N+z)} dz, \tag{20a}$$

$$f_2(N, \delta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(N, z) \Gamma \left(2 - \frac{N-z}{2}\right) (\delta + z - N - 2)^{-\frac{1}{2}(4-N+z)} dz, \tag{20b}$$

$$f_3(N, \delta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(N, z) \Gamma \left(1 - \frac{N-z}{2}\right) (\delta + z - N - 2)^{-\frac{1}{2}(2-N+z)} dz, \tag{20c}$$

where $$\phi(N, z) = \frac{\Gamma(z)(0.169 \exp. \delta)^{N-z+1}}{\Gamma(N+1)\,(N-z+1)^{\frac{3}{2}}}.$$

The values of $f_1(N, \delta)$, $f_2(N, \delta)$, $f_3(N, \delta)$ are to be evaluated by the saddle point method. The accuracy of the saddle point method of integration is well within 2%, as has been tested by the method already mentioned in a previous paper.[7] Taking $\delta = 1.87$, we have evaluated the values of f_1, f_2, f_3 for different values of N, and the values thus obtained are given in Table I. The values of K_0, L_0, $K_{\frac{1}{2}}$, etc., depend also on Z and the ratio (M/m). At the present moment

TABLE I

Values of f_1 (N, δ), f_2 (N, δ), f_3 (N, δ) with $\delta = 1.87$.

N	100	200	400	600	800	1000	1200	1600
$f_1 \times 10^6$	62.99	15.96	4.084	1.876	1.073	.6860	.4890	.2782
$f_2 \times 10^6$	299.8	86.35	24.86	12.02	7.210	4.710	3.393	2.017
$f_3 \times 10^3$		19.05	10.60	7.416	5.840	4.661	4.056	3.117

a certain amount of uncertainty exists as to the value of (M/m). Several experiments seem to suggest that it lies between 150 and 200. Following Bethe[18] (1940) we shall take $(M/m) = 177$. For a given material it is therefore no longer difficult to calculate the values of K_0, L_0, $K_{\frac{1}{2}}$, etc., and also of A. With these values and the values of f_1, f_2, f_3 as given in Table I, the values of B(N), for different values of N and for the different cases mentioned before, can be easily calculated. The results thus obtained have been given in Table II. The figures in the table show that even for N = 1600, the value of B(N) for a spin half meson, when the screening is neglected is less than those when screening is complete. It is therefore clear that even for N = 1600, the meson energy is not high enough for screening to be complete.*

* I am indebted to Dr. Bhabha for drawing my attention to this point in a private communication.

TABLE II

Values of $B(N) \times 10^9$, for different values of N

N→ Spin ↓	100	200	400	600	800	1000	1200	1600
0	47.02	13.41	3.816	1.838	1.144	.7166	.5158	.3057
½ (Screening neglected)	86.66	25.55	7.461	3.631	2.191	1.435	1.035	.6183
½ (Screening complete)	162.1	41.08	10.51	4.829	2.762	1.766	1.259	.7161
1 (Radiation damping neglected)	...	255.5	142.1	99.45	78.31	62.50	54.39	41.80
1 (Radiation damping considered)	478.7	121.3	31.04	14.26	8.155	5.214	3.710	2.114

The frequency of bursts produced through the knock-on process can be similarly calculated. For small bursts these have been obtained by Bhabha, and Bhabha, Carmichael and Chou. For large bursts the knock-on process gives an insignificant contribution to the burst production which can be shown by the following considerations. Let us estimate the relative probability of a shower being produced by the radiation process and the collision (knock-on) process which amounts to estimating the *ratio* R (say) of the chance of a meson emitting a quantum of energy greater than E to the chance of its producing by direct collision an electron of energy greater than the same amount. The cross-sections for these processes have been given in equations (1) and (3). The required ratio is thus given by

$$R = \frac{\int_{E}^{W} Q_r(W, E_0)\,dE_0}{\int_{E}^{E_{0m}} Q_c(W, E_0)\,dE_0} = \frac{I(\text{rad.})}{I(\text{coll.})} \quad \ldots \quad (21)$$

The values of I(rad.) and I(coll.) can be obtained by easy calculations for different spins.

For given values of W and E the values of R can thus be easily calculated. The values of R thus obtained, for different values of W and E, have been given in Table III. The lowest value of E required for a given N is easily obtained from (11) and these values have also been shown in the table III. It shows that as N increases R also increases and for $N \geq 100$, R is much greater than 1,

TABLE III

Values of R for different values of W and E

$R_{\frac{1}{2}}\rightarrow$ spin half and screening complete, $R_1\rightarrow$ spin 1, neglecting the effects of radiation damping, $R_1'\rightarrow$ spin 1, considering the effect of radiation damping.

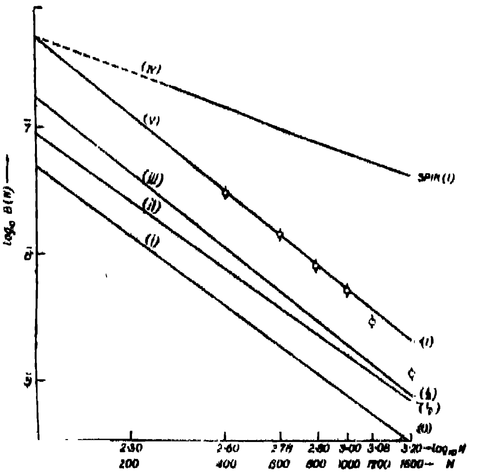
W/Mc^2	E/Mc^2	1.0	5.0	10 0	50 0	100.0
	N~	3	7	12	50	100
	$R_{\frac{1}{2}}$	.1124	.4457	.8760	—	—
10^2	R'_1	0649	.4109	1.064	—	—
	R_1	1.626	10.29	26.64	—	—
	$R_{\frac{1}{2}}$	.1622	.6183	1.082	3 834	6.576
10^3	R'_1	0595	.2948	.5858	2.879	5.827
	R_1	1622	6183	1.082	3.834	6.576
	$R_{\frac{1}{2}}$	2199	8966	1.620	6.165	10.77
10^4	R'_1	.0588	.2817	.5385	2.129	3 469
	R_1	1.473×10^2	7.050×10^2	1.348×10^3	$5\ 329\times10^3$	8.681×10^4

and gradually increases as N increases.* On the other hand for small values of N we get $R<1$, which shows that for small showers the knock-on process predominates while for large bursts the radiation process gives nearly the whole contribution and the knock-on process only producing an insignificant effect. The results obtained by Christy and Kusaka also show this behaviour. The calculations of Bhabha, Charmichael and Chou, in which they considered only the knock-on process, give a good fit with their experiments, since the experiments do not involve sufficiently high energy and the results were obtained in the region where the knock-on process is the predominant one or at least the two processes give equivalent contributions (*i.e.*, $R\sim1$).

The values of B(N) as obtained from equations (17) to (20) can therefore be considered as giving the *total burst frequency* when $N\geq100$. In Fig. 2, B(N) has been plotted as a function of N, on a log-log scale and it is observed that all the cases give practically straight lines. It is observed that the experimental values of Schein and Gill fit very well with the case of *spin 1, when the effect of the radiation damping is considered.* The burst frequencies calculated either with spin 0 or half are very much less than the experimental values. The

* This conclusion is altered if the radiation damping is neglected in the case of collision cross-section for spin 1. It appears that when $E\geq10^{11}$ e.v., the value of R for spin 1 begins to decrease and will ultimately be less than 1. This shows that the radiation damping will also produce important effect on the collision cross-sections, but this effect is perceptible only at very high energies. The values of $R_{\frac{1}{2}}$ when the screening is neglected and also that of R_0 have not been calculated, but their order can easily be obtained from the values of $R_{\frac{1}{2}}$ when the screening is complete. In these cases also the above conclusions hold good.

difference is certainly much more than the possible range of experimental error. The case of spin 1 when the effect of radiation damping is neglected is certainly



Calculated and observed size frequency distribution for large bursts. Numbers (*i*,—(*v*) refer to the different cases given by equations (18), and (ϕ) represents the experimental points of Schein and Gill.

FIG. 2

not a possibility, since both the intensity as well as the slope of the curve in this case is very different from the experimental curve. The experimental point at N = 200 is probably in error since it is possible that some small bursts have escaped detection. For very large bursts again, the statistical errors become very considerable, and the discrepancy at large values of N, say N $\geq$ 1200, is probably due to this.

The only source of uncertainty in the above calculations is the nature of the fluctuations. Unless the actual form for the fluctuation differs considerably from a Poisson's distribution, the results deduced above can be considered to be fairly accurate. The reasons for getting results which are so much different from those of Christy and Kusaka, and consequently leading to just an opposite conclusion are mainly, (*i*) the difference in the form of the fluctuation assumed, (*ii*) the very rough approximation made by them for $\bar{N}$, the average number of particles produced in a cascade shower by an energetic particle or quantum, and (*iii*) the consideration of the effect of radiation damping for spin 1. The consequence of these effects on the calculations will be clear when we compare the values of $J_{N\smile}(y_0)$ as given by (14) with the corresponding expression obtained by Christy and Kusaka,* for various values of N. It may be noted that the validity of the expressions for the cross section for the different processes for very high energies is open to serious doubts as have been pointed out by Oppenheimer,

* The corresponding expression of Christy and Kusaka is $7(E_0/9\beta N)^{\frac{1}{4}}\exp.(-9\beta N/E_0)$ although they have finally modified it and have taken $13.5(E_0/15\beta N)^{\frac{1}{2}}\exp.(-15\beta N/E_0$ in the subsequent calculations.

Snyder and Serber (1940).[19] But similar extension of the range of validity of the quantum-mechanical expressions have been made in many other problems of Cosmic ray phenomena. While making the calculations for the knock-on process, and spin 1, the effect of the radiation damping on this cross-section was not considered. To my knowledge no one has as yet given the form of this cross-section after taking into account the effect of radiation damping, but it appears that this will largely modify the last term in (1c), which gives the spin dependence of the cross-section. But this will not materially affect the conclusions made before, since this term is of importance only for very high energy ($\sim 10^{11}$ e.v.) of the meson, when the total contribution of the knock-on process to the burst production is itself negligible.

We thus see that the frequency-size curve for large bursts provides evidence that the meson cannot have spin 0 or half. The case of spin 1 is a possibility or even a probability only when the effect of radiation damping is taken into consideration. *The meson has therefore a spin of one unit* in agreement with what is believed from nuclear considerations and the above calculations lend further support to the fact that the effect of radiation damping plays an important rôle in the processes involving mesons and the expression given by Wilson for the radiation cross-section is a much more reasonable one than those calculated previously wherein radiation damping was ignored.

APPENDIX

For the case of spin 0, we have from (3a) and (17),

$$\int_{E_0}^{\infty} Q(W, E_0) W^{-(\delta+1)} dW = \frac{16}{3} \cdot \frac{Z^2 r_0^2}{137} \cdot \left(\frac{m}{M}\right)^2$$

$$\times \int_{E_0}^{\infty} \left(\frac{W-E_0}{E_0}\right) \left[\log \frac{2W(W-E_0)}{Mc^2 E_0} - \tfrac{1}{2} \right] W^{-(\delta+2)} dW$$

$$= \frac{16}{3} \frac{Z^2 r_0^2}{137} \cdot \left(\frac{m}{M}\right)^2 E_0^{-(\delta+1)} \left[\delta^{-1} (\delta+1)^{-1} \left\{ \log \frac{2E_0}{Mc^2} - \tfrac{1}{2} \right\} \right.$$

$$\left. - \int_0^1 (1-x) x^{\delta-1} \left\{ \log \frac{x^2}{1-x} \right\} dx \right]$$

$$= \frac{Z^2 r_0^2}{137} \cdot \left(\frac{m}{M}\right)^2 E_0^{-(\delta+1)} \left[K_0 + L_0 \log \frac{E_0}{\beta} \right].$$

$$\therefore \int_{\beta_e \zeta}^{\infty} (y_0 - \zeta)^{-\frac{1}{2}(N-z)} \exp.(N-z+1) y_0 \, dE_0 \int_{E_0}^{\infty} Q(W, E_0) \, W^{-(\delta+1)} dW$$

$$=\frac{Z^2 r_0^2}{137}\left(\frac{m}{M}\right)^2 \beta^{-\delta} \exp.(N-\delta-z-1)\zeta \int_0^\infty x^{-\frac{1}{2}(N-z)} \exp.(N-\delta-z+1)x$$

$$\times\{K_0+L_0\zeta+L_0 x\}dx$$

$$=\frac{Z^2 r_0^2}{137}\left(\frac{m}{M}\right)^2 \beta^{-\delta} \exp.(N-\delta-z-1)\zeta\left[\frac{(K_0+L_0\zeta)\,\Gamma(1-(N-z)/2)}{(\delta+z-N-1)^{(1-(N-z)/2)}}\right.$$

$$\left.+L_0\frac{\Gamma(2-(N-z)/2)}{(\delta-N+z-1)^{(2-(N-z)/2)}}\right]$$

$$\therefore\quad B(N)=A(K_0+\zeta L_0)\exp.(-\delta\zeta)\,.\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi(N,z)\,\Gamma(1-(N-z)/2)$$

$$\times(\delta-N+z-1)^{-(1-(N-z)/2)}.dz.$$

$$+AL_0\exp.(-\delta\zeta).\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi(N,z)\,\Gamma(2-(N-z)/2)$$

$$\times(\delta-N+z-1)^{-(2-(N-z)/2)}.dz.$$

For the case of spin 1, and neglecting the effect of radiation damping,

$$\int_{E_0}^\infty Q(W,E_0)\,W^{-(\delta+1)}dW=\frac{1}{12}.\frac{Z^2 r_0^2}{137}\left(\frac{m}{M}\right)^2.\frac{E_0^{-\delta}}{Mc^2}.$$

$$\times\{2\delta^{-1}(\delta+1)^{-1}+7(\delta+2)^{-1}\}.$$

$$\therefore\int_{\beta e^\zeta}^\infty\frac{\exp.(N-z+1)y_0}{(y_0-\zeta)^{\frac{1}{2}(N-z)}}\,dE_0\,.\int_{E_0}^\infty Q(W,E_0)W^{-(\delta+1)}dW$$

$$=\frac{1}{12}.\frac{Z^2 r_0^2}{137}\left(\frac{m}{M}\right)^2.\frac{\beta^{-(\delta-1)}}{Mc^2}\{2\delta^{-1}(\delta+1)^{-1}+7(\delta+2)^{-1}\}.\exp.\{-(\delta+z-N-2)\zeta\}$$

$$\times\Gamma(1-(N-z)/2)(\delta+z-N-2)^{-(1-(N-z)/2)}$$

$$\therefore\quad B(N)=AK_1\exp.-(\delta-1)\zeta.\frac{1}{2\pi i}.\int_{c-i\infty}^{c+i\infty}\phi(N,z),$$

$$\times\Gamma(1-(N-z)/2)\,(\delta+z-N-2)^{-(1-(N-z)/2)}.dz.$$

With other values of $Q(W, E_0)$ the integrals occurring in (17) may be similarly evaluated and we get the results given in equations (18)—(20).

DEPARTMENT OF APPLIED MATHEMATICS,
CALCUTTA UNIVERSITY.

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